EJERCICIOS DE INTEGRALES.

1. $\int x^5 dx = \frac{x^{5+1}}{5+1} + k = \frac{x^6}{6} + k$

2. $\int (x + \sqrt{x})dx$ $\qquad$ $Sol: \frac{x^2}{2} + \frac{2x\sqrt{x}}{3} + k$

3. $\int \left(\frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{4} \right) dx$ $\qquad$ Sol: $6\sqrt{x} - \frac{1}{10} x^2 \sqrt{x} + k$

4. $\int \frac{x^2 dx}{\sqrt{x}}$ $\qquad$ Sol: $\frac{2}{5} x^2 \sqrt{x} + k$

5. $\int \left(\frac{1}{x^2} + \frac{4}{x\sqrt{x}} + 2 \right) dx$ $\qquad$ Sol: $-\frac{1}{x} - \frac{8}{\sqrt{x}} + 2x + k$

6. $\int \frac{dx}{\sqrt[4]{x}}$ $\qquad$ Sol: $\frac{4}{3} \sqrt[4]{x^3} + k$

7. $\int \left(x^2 + \frac{1}{\sqrt[3]{x}} \right)^2 dx$ $\qquad$ Sol: $\frac{x^5}{5} + \frac{3}{4} x^2 \sqrt[3]{x^2} + 3\sqrt[3]{x} + k$

8. $\int \frac{L\,x}{x} \cdot dx$ $\qquad$ Sol: $\frac{1}{2} L^2 x + k$

9. $\int \text{tg}\, x \cdot \sec^2 x \cdot dx$ $\qquad$ Sol: $\frac{1}{2} \cdot \text{tg}^2\, x + k$

10. $\int \text{sen}^2\, x \cdot \cos x \cdot dx$ $\qquad$ Sol: $\frac{\text{sen}^3\, x}{3} + k$

11. $\int \cos^3 x \cdot \text{sen}\, x \cdot dx$ $\qquad$ Sol: $-\frac{\cos^4 x}{4} + k$

12. $\int x\sqrt{x^2 + 1} \cdot dx$ $\qquad$ Sol: $\frac{1}{3} \sqrt{(x^2 + 1)^3} + k$

13. $\int \frac{x dx}{\sqrt{2x^2 + 3}}$ $\qquad$ Sol: $\frac{1}{2} \sqrt{2x^2 + 3} + k$

14. $\int \frac{x^2 dx}{\sqrt{x^3 + 1}}$ $\qquad$ Sol: $\frac{2}{3} \sqrt{x^3 + 1} + k$

15. $\int \frac{\cos x}{\text{sen}^2\, x} dx$ $\qquad$ Sol: $-\frac{1}{\text{sen}\, x} + k$

16. $\int x\cdot(x^2+1)^4\,dx$ Sol: $\dfrac{(x^2+1)^5}{10}+k$

17. $\int \dfrac{\operatorname{sen} x}{\cos^3 x}\cdot dx$ Sol: $\dfrac{1}{2\cos^2 x}+k$

18. $\int \dfrac{\operatorname{tg} x}{\cos^2 x}\cdot dx$ Sol: $\dfrac{\operatorname{tg}^2 x}{2}+k$

19. $\int \dfrac{\operatorname{cotg} x}{\operatorname{sen}^2 x}\cdot dx$ Sol: $-\dfrac{\operatorname{cotg}^2 x}{2}+k$

20. $\int \dfrac{1}{\cos^2 x\sqrt{\operatorname{tg} x-1}}\cdot dx$ Sol: $2\sqrt{\operatorname{tg} x-1}+k$

21. $\int \dfrac{\mathrm{L}\,(x+1)}{x+1}dx$ Sol: $\dfrac{\mathrm{L}^2\,(x+1)}{2}+k$

22. $\int \dfrac{\cos x}{\sqrt{2\operatorname{sen} x+1}}dx$ Sol: $\sqrt{2\operatorname{sen} x+1}+k$

23. $\int \dfrac{\operatorname{sen} 2x}{(1+\cos 2x)^2}dx$ Sol: $\dfrac{1}{2(1+\cos 2x)}+k$

24. $\int \dfrac{\operatorname{sen} 2x}{\sqrt{1+\operatorname{sen}^2 x}}dx$ Sol: $2\sqrt{1+\operatorname{sen}^2 x}+k$

25. $\int \dfrac{\sqrt{\operatorname{tg} x+1}}{\cos^2 x}\cdot dx$ Sol: $\dfrac{2}{3}\sqrt{(\operatorname{tg} x+1)^3}+k$

26. $\int \dfrac{\cos 2x}{(2+3\operatorname{sen} 2x)^3}dx$ Sol: $-\dfrac{1}{12}\dfrac{1}{(2+3\operatorname{sen} 2x)^2}+k$

27. $\int \dfrac{\operatorname{sen} 3x}{\sqrt[3]{\cos^4 3x}}dx$ Sol: $\dfrac{1}{\sqrt[3]{\cos 3x}}+k$

28. $\int \dfrac{\mathrm{Ln}^2 x\ \ dx}{x}$ Sol: $\dfrac{\mathrm{Ln}^3 x}{3}+k$

29. $\int \dfrac{\operatorname{arc\,sen} x\,dx}{\sqrt{1-x^2}}$ Sol: $\dfrac{\operatorname{arc\,sen}^2 x}{2}+k$

30. $\int \dfrac{\operatorname{arc\,cos}^2 x\,dx}{\sqrt{1-x^2}}$ Sol: $-\dfrac{\operatorname{arc\,cos}^3 x}{3}+k$

31. $\int \frac{\operatorname{arc\,tg} x}{1+x^2}\,dx$ Sol: $\frac{\operatorname{arc\,tg}^2 x}{2}+k$

32. $\int \frac{\operatorname{arc\,ctg} x}{1+x^2}\,dx$ Sol: $-\frac{\operatorname{arc\,ctg}^2 x}{2}+k$

33. $\int \frac{x}{x^2+1}\,dx$ Sol: $\frac{1}{2}\operatorname{Ln}(x^2+1)+k$

34. $\int \frac{dx}{1-x}$ Sol: $-\operatorname{Ln}|1-x|+k$

35. $\int \frac{dx}{3x-7}$ Sol: $\frac{1}{3}\operatorname{L}|3x-7|+k$

36. $\int \frac{dx}{5-2x}$ Sol: $-\frac{1}{2}\operatorname{L}|5-2x|+k$

37. $\int \frac{x+1}{x^2+2x+3}\,dx$ Sol: $\frac{1}{2}\operatorname{L}\left|x^2+2x+3\right|+k$

38. $\int \frac{dx}{x\cdot \operatorname{Ln} x}$ Sol: $\operatorname{Ln}|\operatorname{Ln} x|+k$

39. $\int \operatorname{tg} x\,dx$ Sol: $-\operatorname{Ln}|\cos x|+k$

40. $\int \operatorname{tg} 2x\,dx$ Sol: $-\frac{1}{2}\operatorname{L}|\cos 2x|+k$

41. $\int \operatorname{ctg} x\,dx$ Sol: $\operatorname{Ln}|\operatorname{sen} x|+k$

42. $\int \operatorname{ctg}(5x-7)\,dx$ Sol: $\frac{1}{5}\operatorname{L}|\operatorname{sen}(5x-7)|+k$

43. $\int \frac{dx}{\operatorname{ctg} 3x}$ Sol: $-\frac{1}{3}\operatorname{L}|\cos 3x|+k$

44. $\int \operatorname{ctg}\frac{x}{3}\,dx$ Sol: $3\operatorname{L}\left|\operatorname{sen}\frac{x}{3}\right|+k$

45. $\int (\operatorname{ctg} e^x)e^x\,dx$ Sol: $\operatorname{L}\left|\operatorname{sen} e^x\right|+k$

46. $\int \left(\operatorname{tg} 4x-\operatorname{ctg}\frac{x}{4}\right)dx$ Sol: $\text{Sol}:-\frac{1}{4}\operatorname{Ln}|\cos 4x|-4\operatorname{Ln}\left|\operatorname{sen}\frac{x}{4}\right|+k$

47. $\int \frac{\cos x}{2\operatorname{sen} x+3}dx$ Sol: $\frac{1}{2}\operatorname{Ln}(2\operatorname{sen} x+3)+k$

48. $\int \frac{dx}{(1+x^2)\operatorname{arc\,tg} x}$ Sol: $\operatorname{Ln}\left|\operatorname{arc\,tg} x\right|+k$

49. $\int \frac{dx}{\cos^2 x(3\operatorname{tg} x+1)}$ Sol: $\frac{1}{3}\operatorname{Ln}(3\operatorname{tg} x+1)+k$

50. $\int \frac{dx}{\sqrt{1-x^2}\operatorname{arc\,sen} x}$ Sol: $\operatorname{Ln}\left|\operatorname{arc\,sen} x\right|+k$

51. $\int \frac{\cos 2x}{2+3\operatorname{sen} 2x}dx$ Sol: $\frac{1}{6}\operatorname{Ln}\left|2+3\operatorname{sen} 2x\right|+k$

52. $\int e^{2x}dx$ Sol: $\frac{1}{2}e^{2x}+k$

53. $\int e^{\frac{x}{2}}dx$ Sol: $2e^{\frac{x}{2}}+k$

54. $\int e^{\operatorname{sen} x}\cos x\,dx$ Sol: $e^{\operatorname{sen} x}+k$

55. $\int a^{x^2}\cdot x\cdot dx$ Sol: $\frac{a^{x^2}}{2\operatorname{L} a}+k$

56. $\int e^{\frac{x}{a}}dx$ Sol: $ae^{\frac{x}{a}}+k$

57. $\int \left(e^{2x}\right)^2 dx$ Sol: $\frac{1}{4}e^{4x}+k$

58. $\int e^{-3x}dx$ Sol: $-\frac{1}{3}e^{-3x}+k$

59. $\int 5^x e^x dx$ Sol: $\frac{5^x e^x}{\operatorname{Ln} 5+1}+k$

60. $\int \left(e^{5x}+a^{5x}\right)dx$ Sol: $\frac{1}{5}\left(e^{5x}+\frac{a^{5x}}{\operatorname{L} a}\right)+k$

61. $\int e^{x^2+4x+3}(x+2)dx$ Sol: $\frac{1}{2}e^{x^2+4x+3}+k$

62. $\int \frac{e^x}{3+4e^x}dx$ Sol: $\frac{1}{4}\text{Ln}(3+4e^x)+k$

63. $\int \cos 5x dx$ Sol: $\frac{1}{5}\text{sen}\,5x+k$

64. $\int \text{sen}\frac{x}{3}dx$ Sol: $-3\cos\frac{x}{3}+k$

65. $\int \sec^2(7x+2)dx$ Sol: $\frac{1}{7}\text{tg}(7x+2)+k$

66. $\int x\cos 3x^2 dx$ Sol: $\frac{1}{6}\text{sen}\,3x^2+k$

67. $\int \text{tg}^2 x\,dx$ Sol: $\text{tg}\,x-x+k$

Por trigonometría sabemos que $\text{tg}^2 x+1=\sec^2 x \quad \Rightarrow \quad \text{tg}^2 x=\sec^2 x-1,$ entonces

$$\int \text{tg}^2 x\,dx=\int(\sec^2 x-1)dx=\int \sec^2 x\,dx-\int dx=\text{tg}\,x-x+k$$

68. $\int \frac{\cos(\text{Ln}(x))}{x}dx$ Sol: $\text{sen}\,(\text{Ln}(x))+k$

69. $\int \text{tg}^3 x\,dx$ Sol: $\frac{\text{tg}^2 x}{2}+\text{Ln}\left|\cos x\right|+k$

$$\int \text{tg}^3 x\,dx=\int \text{tg}\,x\cdot\text{tg}^2 x\,dx=\int \text{tg}\,x\cdot(\sec^2 x-1)\,dx=\int \underbrace{\text{tg}\,x}_{f^1}\cdot\underbrace{\sec^2 x}_{f}\,dx-\int \text{tg}\,x\,dx=$$

$$=\frac{\text{tg}^2 x}{2}-\int\frac{\text{sen}\,x}{\cos x}dx=\frac{\text{tg}^2 x}{2}+\int\frac{-\text{sen}\,x}{\cos x}dx=\frac{\text{tg}^2 x}{2}+\text{Ln}\left|\cos x\right|+k$$

70. $\int \cos\sqrt{x}\,\frac{dx}{\sqrt{x}}$ Sol: $2\,\text{sen}\sqrt{x}+k$

71. $\int \frac{x}{\sqrt{1-x^4}}dx$ Sol: $\frac{1}{2}\text{arc sen}\,x^2+k$

$$\int\frac{x}{\sqrt{1-x^4}}dx=\int\frac{x}{\sqrt{1-(x^2)^2}}dx=\left\{\int\frac{f'(x)}{\sqrt{1-(f(x))^2}}dx=\begin{cases}\text{arc sen}\,f(x)+k\\-\arccos f(x)+k\end{cases}\right\}=$$

$$=\frac{1}{2}\int\frac{2x}{\sqrt{1-(x^2)^2}}dx=\left\{\int\frac{f'(x)dx}{\sqrt{1-(f(x))^2}}\right\}=\frac{1}{2}\operatorname{arc sen}(x^2)+k$$

72. $\int\frac{dx}{\sqrt{1-4x^2}}$ Sol: $\frac{1}{2}\operatorname{arc sen}(2x)+k$

73. $\int\frac{dx}{\sqrt{9-4x^2}}$ Sol: $\frac{1}{2}\operatorname{arc sen}(\frac{2x}{3})+k$

$$\int\frac{dx}{\sqrt{9-4x^2}}=\int\frac{dx}{\sqrt{9(1-\frac{4x^2}{9})}}=\int\frac{dx}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}}=\frac{1}{3}\int\frac{dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}=\frac{1}{3}\cdot\frac{3}{2}\int\frac{\frac{2}{3}dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}=$$

$$=\frac{1}{2}\int\frac{\frac{2}{3}dx}{\sqrt{1-\left(\frac{2x}{3}\right)^2}}=\left\{\int\frac{f'(x)dx}{\sqrt{1-(f(x))^2}}\right\}=\frac{1}{2}\operatorname{arc sen}(\frac{2x}{3})+k$$

74. $\int\frac{e^x}{3+4e^x}dx$ Sol: $\frac{1}{4}\operatorname{Ln}(3+4e^x)+k$

75. $\int\frac{e^{2x}}{2+e^{2x}}dx$ Sol: $\frac{1}{2}\operatorname{Ln}(2+e^{2x})+k$

76. $\int\frac{e^x}{1+e^{2x}}dx$ Sol: $\operatorname{arc tg}(e^x)+k$

77. $\int\frac{dx}{1+2x^2}$ Sol: $\frac{1}{\sqrt{2}}\operatorname{arc tg}(\sqrt{2}x)+k$

78. $\int\frac{dx}{4+x^2}$ Sol: $\frac{1}{2}\operatorname{arc tg}(\frac{x}{2})+k$

79. $\int\frac{dx}{x\sqrt{1-\operatorname{Ln}^2(x)}}$ Sol: $\operatorname{arc sen}(\operatorname{Ln}(x))+k$

80. $\int\frac{\arccos x-x}{\sqrt{1-x^2}}dx$ Sol: $-\frac{1}{2}(\arccos(x))^2+\sqrt{1-x^2}+k$

81. $\int\frac{x-\operatorname{arctg}x}{1+x^2}dx$ Sol: $\frac{1}{2}\operatorname{Ln}(1+x^2)-\frac{1}{2}(\operatorname{arctg}x)^2+k$

82. $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ Sol: $\dfrac{4}{3}\sqrt{(1+\sqrt{x})^3}+k$

83. $\displaystyle\int \frac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}}\,dx$ Sol: $4\sqrt{1+\sqrt{x}}+k$

84. $\displaystyle\int x\sqrt{x-1}\,dx$ Sol: $\dfrac{2}{5}(x-1)^{\frac{5}{2}}+\dfrac{2}{3}(x-1)^{\frac{3}{2}}+k$

85. $\displaystyle\int x(5x^2-3)^7\,dx$ Sol: $\dfrac{1}{80}(5x^2-3)^8+k$

86. $\displaystyle\int x(2x+5)^{10}\,dx$ Sol: $\dfrac{1}{4}\left[\dfrac{(2x+5)^{12}}{12}-\dfrac{5(2x+5)^{11}}{11}\right]+k$

87. $\displaystyle\int xe^x\,dx$ Sol: $e^x(x-1)+k$

88. $\displaystyle\int (x^2-3x+5)e^x\,dx$ Sol: $e^x(x^2-5x+10)+k$

89. $\displaystyle\int x\,\mathrm{Ln}(x)\,dx$ Sol: $\dfrac{1}{2}x^2\left(\mathrm{Ln}(x)-\dfrac{1}{2}\right)+k$

Por el método de integración por partes:

$$\int x\,\mathrm{Ln}(x)\,dx=\left\{\begin{array}{l} u=\mathrm{Ln}(x)\Rightarrow du=\dfrac{1}{x}dx \\ dv=xdx\Rightarrow v=\dfrac{1}{2}x^2 \end{array}\right\}=\frac{1}{2}x^2\,\mathrm{Ln}(x)-\int\frac{1}{2}x^2\cdot\frac{1}{x}dx=$$

$$=\frac{1}{2}x^2\,\mathrm{Ln}(x)-\frac{1}{2}\int x\,dx=\frac{1}{2}x^2\,\mathrm{Ln}(x)-\frac{1}{2}\cdot\frac{1}{2}x^2+k=\frac{1}{2}x^2\left(\mathrm{Ln}(x)-\frac{1}{2}\right)+k$$

90. $\displaystyle\int \mathrm{Ln}(x)\,dx$ Sol: $x(\mathrm{Ln}(x)-1)+k$

91. $\displaystyle\int x\,\mathrm{sen}\,x\,dx$ Sol: $\mathrm{sen}\,x-x\cos x+k$

92. $\displaystyle\int x\cos^2 x\,dx$ Sol: $\dfrac{x^2}{4}+\dfrac{1}{4}x\,\mathrm{sen}\,2x+\dfrac{1}{8}\cos 2x+k$

93. $\displaystyle\int e^{-x}\cos x\,dx$ Sol: $\dfrac{1}{2}e^{-x}(\mathrm{sen}\,x-\cos x)+k$

94. $\int \mathrm{Ln}(1-x)\,dx$ Sol: $-x-(1-x)\mathrm{Ln}(1-x)+k$

$$\int \mathrm{Ln}(1-x)\,dx = \left\{ \begin{array}{c} u=\mathrm{Ln}(1-x) \Rightarrow du = \dfrac{-1}{1-x}dx \\ dv = dx \Rightarrow v = x \end{array} \right\} = x\mathrm{Ln}(1-x) - \int x \cdot \frac{-1}{1-x}dx =$$

$$= x\mathrm{Ln}(1-x) - \int \frac{-x}{1-x}dx = x\mathrm{Ln}(1-x) - \int \frac{1-x-1}{1-x}dx = x\mathrm{Ln}(1-x) - \int \left(1+\frac{-1}{1-x}\right)dx =$$

$$= x\mathrm{Ln}(1-x) - \left(x+\mathrm{Ln}(1-x)\right) + k = x\mathrm{Ln}(1-x) - x - \mathrm{Ln}(1-x) + k =$$

$$= -x-(1-x)\mathrm{Ln}(1-x)+k$$

95. $\int \arcsin x\,dx$ Sol: $x\arcsin x + \sqrt{1-x^2} + k$

96. $\int \sqrt{1-x^2}\,dx$ Sol: $\frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)+k$

97. $\int x\arcsin x\,dx$ Sol: $\frac{1}{4}\left[(2x^2-1)\arcsin x + x\sqrt{1-x^2}\right]+k$

98. $\int \mathrm{arc\,tg}\, x\,dx$ Sol: $x\,\mathrm{arc\,tg}\,x - \frac{1}{2}\mathrm{Ln}(1+x^2)+k$

99. $\int \mathrm{arc\,tg}\sqrt{x}\,dx$ Sol: $(x+1)\,\mathrm{arc\,tg}\sqrt{x} - \sqrt{x} + k$

100. $\int \frac{x\arcsin x}{\sqrt{1-x^2}}dx$ Sol: $x - \sqrt{1-x^2}\arcsin x + k$

101. $\int \frac{2x-1}{(x-1)(x-2)}dx$ Sol: $\mathrm{Ln}\left|\frac{(x-2)^3}{x-1}\right|+k$

Tenemos una integral de tipo racional donde el grado del numerador es menor que el grado del denominador. Vamos a descomponer el integrando en fracciones simples:

$$(x-1)(x-2)=0 \Rightarrow \left\{ \begin{array}{l} x=1 \\ x=2 \end{array} \right. \text{ (raíces reales simples)}$$

Entonces:

$$\frac{2x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Vamos a calcular los coeficientes indeterminados. Al ser los denominadores iguales, los numeradores también lo serán. Por tanto:

$$2x-1=A(x-2)+B(x-1)\Rightarrow\begin{cases}x=1\rightarrow 1=-A\Rightarrow A=-1\\ x=2\rightarrow 3=B\ \ \Rightarrow B=3\end{cases}$$

Por tanto,

$$\int\frac{2x-1}{(x-1)(x-2)}dx=\int\left(\frac{-1}{x-1}+\frac{3}{x-2}\right)dx=-\int\frac{1}{x-1}dx+3\int\frac{1}{x-2}dx=$$

$$=-\text{Ln }(x-1)+3\ \ \text{Ln }(x-2)+k=\text{Ln}\left|\frac{(x-2)^3}{x-1}\right|+k$$

102. $\displaystyle\int\frac{xdx}{(x+1)(x+3)(x+5)}$ Sol: $\displaystyle\frac{1}{8}\ \text{Ln}\left|\frac{(x+3)^6}{(x+1)(x+5)^5}\right|+k$

Tenemos una integral de tipo racional donde el grado del numerador es menor que el grado del denominador. Vamos a descomponer el integrando en fracciones simples:

$$(x+1)(x+3)(x+5)=0\Rightarrow\begin{cases}x=-1\\ x=-3\\ x=-5\end{cases}\quad\text{(raíces reales simples)}$$

Entonces:

$$\frac{x}{(x+1)(x+3)(x+5}=\frac{A}{x+1}+\frac{B}{x+3}+\frac{C}{x+5}=$$

$$=\frac{A(x+3)(x+5)+B(x+1)(x+5)+C(x+1)(x+3)}{(x+1)(x+3)(x+5)}$$

Para calcular los coeficientes indeterminados, al ser los denominadores iguales, los numeradores también lo serán. Por tanto:

$$x=A(x+3)(x+5)+B(x+1)(x+5)+C(x+1)(x+3)\Rightarrow\begin{cases}x=-1\rightarrow -1=8A\Rightarrow A=-\frac{1}{8}\\ x=-3\rightarrow -3=-4B\ \ \Rightarrow B=\frac{3}{4}\\ x=-5\rightarrow -5=8C\Rightarrow C=-\frac{5}{8}\end{cases}$$

Por tanto,

$$\int \frac{xdx}{(x+1)(x+3)(x+5)} = \int \left(\frac{-\frac{1}{8}}{x+1} + \frac{\frac{3}{4}}{x+3} + \frac{-\frac{5}{8}}{x+5} \right) dx =$$

$$= -\frac{1}{8}\int \frac{1}{x+1}dx + \frac{3}{4}\int \frac{1}{x+3}dx - \frac{5}{8}\int \frac{1}{x+5}dx = -\frac{1}{8}\mathrm{Ln}(x+1) + \frac{3}{4}\mathrm{Ln}(x+3) - \frac{5}{8}\mathrm{Ln}(x+5) + k =$$

$$= \frac{1}{8}\left(\mathrm{Ln}\,(x+1) + 6\,\mathrm{Ln}\,(x+3) - 5\,\mathrm{Ln}\,(x+5)\right) + k = \frac{1}{8}\mathrm{Ln}\left| \frac{(x+3)^6}{(x+1)(x+5)^5} \right| + k$$

103. $\displaystyle\int \frac{x^5 + x^4 - 8}{x^3 - 4x}dx$ Sol: $\displaystyle\frac{x^3}{3} + \frac{x^2}{2} + 4x + \mathrm{Ln}\left| \frac{x^2(x-2)^5}{(x+2)^3} \right| + k$

Al ser el grado del numerador mayor que el grado del denominador, antes de aplicar el método de descomposición en fracciones simples tendremos que dividir. De esta forma obtenemos:

$$\frac{x^5 + x^4 - 8}{x^3 - 4x} = x^2 + x + 4 + \frac{4x^2 + 16x - 8}{x^3 - 4x}$$

En consecuencia:

$$\int \frac{x^5 + x^4 - 8}{x^3 - 4x} \cdot dx = \int (x^2 + x + 4)dx + \int \frac{4x^2 + 16x - 8}{x^3 - 4x} \cdot dx =$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 4x + \int \frac{4x^2 + 16x - 8}{x^3 - 4x} \cdot dx$$

A la integral que nos queda le aplicamos el método de descomposición en fracciones simples. Calculamos las raíces del denominador:

$$x^3 - 4x = 0 \quad \rightarrow \quad x \cdot (x^2 - 4) = 0 \quad \rightarrow \quad \begin{cases} x = 0 \\ x = \pm 2 \end{cases}$$

Entonces:

$$\frac{4x^2 + 16x - 8}{x^3 - 4x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} = \frac{A(x-2)(x+2) + Bx(x+2) + Cx(x-2)}{x(x-2)(x+2)}$$

Como los denominadores son iguales, los numeradores también lo serán; por tanto:

$$4x^2 + 16x - 8 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$$

Calculamos los coeficientes indeterminados: le vamos asignando los valores de las raíces

$$\begin{array}{lllll} x=0 & \rightarrow & -8=-4A & \rightarrow & A=2 \\ x=2 & \rightarrow & 40=\ \ 8B & \rightarrow & B=5 \\ x=-2 & \rightarrow & -24=8C & \rightarrow & C=-3 \end{array}$$

Por tanto, la fracción descompuesta en fracciones simples nos queda:

$$\frac{4x^2+16x-8}{x^3-4x}=\frac{2}{x}+\frac{5}{x-2}-\frac{3}{x+2}$$

La integral de la función pedida será:

$$\int\frac{x^5+x^4-8}{x^3-4x}\cdot dx=\frac{x^3}{3}+\frac{x^2}{2}+4x+\int\frac{4x^2+16x-8}{x^3-4x}\cdot dx=$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+\int\left(\frac{2}{x}+\frac{5}{x-2}-\frac{3}{x+2}\right)\cdot dx$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+\int\frac{2}{x}\cdot dx+\int\frac{5}{x-2}\cdot dx-\int\frac{3}{x+2}\cdot dx=$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+2\int\frac{1}{x}\cdot dx+5\int\frac{1}{x-2}\cdot dx-3\int\frac{1}{x+2}\cdot dx=$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+2\cdot \mathrm{Ln}\,|x|+5\cdot \mathrm{Ln}\,|x-2|-3\cdot \mathrm{Ln}\,|x+2|+k=$$

$$=\frac{x^3}{3}+\frac{x^2}{2}+4x+\mathrm{Ln}\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|+k$$

104. $\displaystyle\int\frac{x^4dx}{(x^2-1)(x+2)}$ Sol: $\displaystyle\frac{x^2}{2}-2x+\frac{1}{6}\ \mathrm{Ln}\left|\frac{(x-1)}{(x+1)^3}\right|+\frac{16}{3}\cdot \mathrm{Ln}\,|x+2|+k$

Como el grado del numerador es mayor que el del denominador, tenemos que dividir, obteniendo:

$$\frac{x^4}{(x^2-1)(x+2)}=x-2+\frac{5x^2-4}{(x^2-1)(x+2)}$$

Con lo que

$$\int\frac{x^4\cdot dx}{(x^2-1)(x+2)}=\int(x-2)\,dx+\int\frac{5x^2-4}{(x^2-1)(x+2)}dx=\frac{x^2}{2}-2x+\int\frac{5x^2-4}{(x^2-1)(x+2)}dx$$

y tendremos que integrar la función racional que nos queda, donde el grado del numerador es menor que el grado del denominador.

Descomponemos en fracciones simples:

$$\frac{5x^2-4}{(x^2-1)(x+2)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{x+2}=\frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{(x^2-1)(x+2)}$$

Como los denominadores son iguales, también lo serán los numeradores. Entonces:

$$5x^2-4=A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)$$

Calculamos los coeficientes indeterminados:

$$\begin{array}{lllll} x=1 & \to & 1=6A & \to & A=\frac{1}{6} \\ x=-1 & \to & 1=-2B & \to & B=-\frac{1}{2} \\ x=-2 & \to & 16=3C & \to & C=\frac{16}{3} \end{array}$$

Entonces: $$\frac{5x^2-4}{(x^2-1)(x+2)}=\frac{\frac{1}{6}}{x-1}+\frac{-\frac{1}{2}}{x+1}+\frac{\frac{16}{3}}{x+2}$$

Y, por tanto:

$$\int\frac{x^4\cdot dx}{(x^2-1)(x+2)}=\frac{x^2}{2}-2x+\int\frac{5x^2-4}{(x^2-1)(x+2)}dx=\frac{x^2}{2}-2x+\int\left(\frac{\frac{1}{6}}{x-1}+\frac{-\frac{1}{2}}{x+1}+\frac{\frac{16}{3}}{x+2}\right)\cdot dx=$$

$$=\frac{x^2}{2}-2x+\int\frac{\frac{1}{6}}{x-1}\cdot dx+\int\frac{-\frac{1}{2}}{x+1}\cdot dx+\int\frac{\frac{16}{3}}{x+2}\cdot dx=$$

$$=\frac{x^2}{2}-2x+\frac{1}{6}\int\frac{1}{x-1}\cdot dx-\frac{1}{2}\int\frac{1}{x+1}\cdot dx+\frac{16}{3}\int\frac{1}{x+2}\cdot dx=$$

$$=\frac{x^2}{2}-2x+\frac{1}{6}\cdot \text{Ln}\,|x-1|-\frac{1}{2}\cdot \text{Ln}\,|x+1|+\frac{16}{3}\cdot \text{Ln}\,|x+2|+k=$$

$$=\frac{x^2}{2}-2x+\frac{1}{6}\cdot\left(\text{Ln}\,|x-1|-3\cdot \text{Ln}\,|x+1|\right)+\frac{16}{3}\cdot \text{Ln}\,|x+2|+k=$$

$$=\frac{x^2}{2}-2x+\frac{1}{6}\cdot \text{Ln}\left|\frac{x-1}{(x+1)^3}\right|+\frac{16}{3}\cdot \text{Ln}\,|\,x+2\,|+k$$

105. $\displaystyle\int \frac{dx}{(x-1)^2(x-2)}$ Sol: $\displaystyle\frac{1}{x-1}+\text{Ln}\left|\frac{x-2}{x-1}\right|+k$

Como el grado del numerador es menor que el grado del denominador aplicamos la descomposición en fracciones simples directamente:

$$\frac{1}{(x-1)^2(x-2)}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-2)}=\frac{A(x-1)(x-2)+B(x-2)+C(x-1)^2}{(x-1)^2(x-2)} \quad \rightarrow$$

$$\rightarrow \quad 1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$$

Calculamos los coeficientes:

$x=1: \quad 1=-B \quad \rightarrow \quad B=-1$
$x=2: \quad 1=C$
$x=0: \quad 1=2A-2B+C \quad \rightarrow \quad 1=2A+2+1 \quad \rightarrow \quad A=-1$

Entonces:

$$\int \frac{1}{(x-1)^2(x-2)}\cdot dx=\int \frac{-1}{x-1}\cdot dx+\int \frac{-1}{(x-1)^2}\cdot dx+\int \frac{1}{(x-2)}\cdot dx=$$

$$=-\int \frac{1}{x-1}\cdot dx-\int (x-1)^{-2}\cdot dx+\int \frac{1}{x-2}\cdot dx=$$

$$=-\text{Ln}\,|\,x-1\,|-\frac{(x-1)^{-1}}{-1}+\text{Ln}\,|\,x-2\,|+k=\frac{1}{x-1}+\text{Ln}\left|\frac{x-2}{x-1}\right|+k$$

106. $\displaystyle\int \frac{x-8}{x^3-4x^2+4x}dx$ Sol: $\displaystyle\frac{3}{x-2}+\text{Ln}\,\frac{(x-2)^2}{x^2}+k$

Igual que en el anterior, aplicamos la descomposición en fracciones simples:

Calculamos las raíces del denominador:

$$x^3-4x^2+4x=0 \quad \rightarrow \quad x\cdot(x^2-4x+4)=0 \quad \rightarrow \quad x\cdot(x-2)^2=0 \quad \rightarrow \quad \begin{cases} x=0 \\ x=2 \quad \text{(doble)} \end{cases}$$

Entonces: $$\frac{x-8}{x^3-4x^2+4x}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{(x-2)^2}=\frac{A(x-2)^2+Bx(x-2)+Cx}{x(x-2)^2} \quad \Rightarrow$$

$$\Rightarrow \quad x-8=A(x-2)^2+Bx(x-2)+Cx \quad \Rightarrow$$

Calculamos los coeficientes:

$$x=0 \quad \to \quad -8=4A \quad \to \quad A=-2$$
$$x=2 \quad \to \quad -6=2C \quad \to \quad C=-3$$
$$x=1 \quad \to \quad -7=A-B+C \quad \to \quad B=7+A+C=7-2-3=2 \quad \to \quad B=2$$

Entonces: $\displaystyle\int \frac{x-8}{x^3-4x^2+4x}\,dx=\int\left(\frac{-2}{x}+\frac{2}{x-2}+\frac{-3}{(x-2)^2}\right)\cdot dx=$

$$=-2\int\frac{1}{x}\cdot dx+2\int\frac{1}{x-2}\cdot dx-3\int\frac{1}{(x-2)^2}\cdot dx=-2\,\mathrm{Ln}\,|x|+2\,\mathrm{Ln}\,|x-2|-3\int(x-2)^{-2}dx=$$

$$=-2\,\mathrm{Ln}\,|x|+2\,\mathrm{Ln}\,|x-2|-3\cdot\frac{(x-2)^{-1}}{-1}+k=\frac{3}{x-2}+\mathrm{Ln}\,\frac{(x-2)^2}{x^2}+k$$

107. $\displaystyle\int\frac{3x+2}{x(x+1)^3}\,dx$ Sol: $\displaystyle\frac{4x+3}{2(x+1)^2}+\mathrm{Ln}\,\frac{x^2}{(x+1)^2}+k$

108. $\displaystyle\int\frac{3x-2}{x(x+1)^3}\,dx$ Sol: $\displaystyle\mathrm{Ln}\,\frac{(x+1)^2}{x^2}-\frac{4x+9}{2(x+1)^2}+k$

Descomponemos el integrando en fracciones simples:

$$\frac{3x-2}{x(x+1)^3}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{(x+1)^2}+\frac{D}{(x+1)^3}=\frac{A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx}{x\cdot(x+1)^3} \quad \to$$

$$\to \quad 3x-2=A(x+1)^3+Bx(x+1)^2+Cx(x+1)+Dx \quad \to$$

Calculamos los coeficientes:

$$x=0 \quad \to \quad -2=A$$
$$x=-1 \to \quad -5=-D \quad \to \quad D=5$$
$$x=1 \quad \to \quad 1=8A+4B+2C+D \quad \to \quad 1=-16+4B+2C+5 \quad \to \quad 2B+C=6$$
$$x=-2 \to \quad -8=-A-2B+2C-2D \quad \to \quad -8=2-2B+2C-10 \quad \to \quad -B+C=0$$

Resolviendo el sistema resultante, obtenemos:

$$\left.\begin{array}{l}2B+C=6\\-B+C=0\end{array}\right\} \quad \to \quad \left\{\begin{array}{l}2B+B=6 \quad \to \quad B=2\\C=B=2\end{array}\right.$$

Entonces: $\displaystyle\int\frac{3x-2}{x(x+1)^3}\cdot dx=\int\frac{-2}{x}\cdot dx+\int\frac{2}{x+1}\cdot dx+\int\frac{2}{(x+1)^2}\cdot dx+\int\frac{5}{(x+1)^3}\cdot dx=$

$$=-2\int\frac{1}{x}\cdot dx+2\int\frac{1}{x+1}\cdot dx+2\int\frac{1}{(x+1)^2}\cdot dx+5\int\frac{1}{(x+1)^3}\cdot dx=$$

$$= -2 \cdot \mathrm{Ln}\,|x| + 2 \cdot \mathrm{Ln}\,|x+1| + 2\int (x+1)^{-2} \cdot dx + 5\int (x+1)^{-3} \cdot dx =$$

$$= -2 \cdot \mathrm{Ln}\,|x| + 2 \cdot \mathrm{Ln}\,|x+1| + 2\frac{(x+1)^{-1}}{-1} + 5\frac{(x+1)^{-2}}{-2} + k =$$

$$= \mathrm{Ln}\frac{(x+1)^2}{x^2} - \frac{2}{x+1} - \frac{5}{2(x+1)^2} + k = \mathrm{Ln}\frac{(x+1)^2}{x^2} - \frac{4(x+1)+5}{2(x+1)^2} + k =$$

$$= \mathrm{Ln}\frac{(x+1)^2}{x^2} - \frac{4x+9}{2(x+1)^2} + k$$

109. $\int \frac{x^2 dx}{(x+2)^2 (x+4)^2}$ Sol: $-\frac{5x+12}{x^2+6x+8} + \mathrm{Ln}\left(\frac{x+4}{x+2}\right)^2 + k$

110. $\int \frac{\sqrt{x}}{\sqrt[4]{x^3}+1}dx$ Sol: $\frac{4}{3}\left(\sqrt[4]{x^3} - \mathrm{Ln}\left(\sqrt[4]{x^3}+1\right)\right) + k$

111. $\int \frac{\sqrt{x^3} - \sqrt[3]{x}}{6\sqrt[4]{x}}dx$ Sol: $\frac{2}{27}\sqrt[4]{x^9} - \frac{2}{13}\sqrt[12]{x^{13}} + k$

112. $\int \frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[4]{x^5}}dx$ Sol: $-\frac{6}{\sqrt[6]{x}} + \frac{12}{\sqrt[12]{x}} + 2\,\mathrm{Ln}\,x - 24\,\mathrm{Ln}\,(\sqrt[12]{x}+1) + k$

113. $\int \frac{\sqrt[7]{x}+\sqrt{x}}{\sqrt[7]{x^8}+\sqrt[14]{x^{15}}}dx$ Sol: $4\left[\sqrt[14]{x} - \frac{1}{2}\sqrt[7]{x} + \frac{1}{3}\sqrt[14]{x^3} - \frac{1}{4}\sqrt[7]{x^2} + \frac{1}{5}\sqrt[14]{x^5}\right] + k$

114. $\int \frac{dx}{\sqrt{1-x}+\sqrt[3]{1-x}}$ Sol: $6\left[\frac{\sqrt{1-x}}{3} - \frac{\sqrt[3]{1-x}}{2} + \sqrt[6]{1-x} - Ln\left(1+\sqrt[6]{1-x}\right)\right] + k$

115. $\int \frac{e^x}{e^{2x}+e^x-2}dx$ Sol: $\frac{1}{3}\ \mathrm{Ln}\left|\frac{e^x-1}{e^x+2}\right| + k$

116. $\int \frac{dx}{e^x+1}$ Sol: $x - \mathrm{Ln}\,(e^x+1) + k$

117. $\int \frac{e^x}{e^{2x}+3e^x+2}dx$ Sol: $\mathrm{Ln}\left|\frac{e^x+1}{e^x+2}\right| + k$

118. $\int \mathrm{sen}^3 x\,dx$ Sol: $\frac{1}{3}\cos^3 x - \cos x + k$