EJEMPLO 1. Determinar el valor exacto de sen (105°)

$$\text{sen}(105°) = \text{sen}(60° + 45°) = \text{sen}\,60° \cos 45° + \cos 60° \,\text{sen}\, 45° = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\,\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

EJEMPLO 2. Determinar el valor exacto de cos(15°)

$$\cos(15°) = \cos(45° - 30°) = \cos 45° \cos 30° + \text{sen}\, 45°\, \text{sen}\, 30° = \frac{\sqrt{2}}{2}\,\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\,\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

EJEMPLO 3. Determinar el valor exacto de sen 22,5° y tan 22,5°.

$$\text{sen}\, 22{,}5° = \text{sen}\,\frac{45°}{2} = \pm\sqrt{\frac{1-\cos 45°}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\tan 22{,}5° = \tan\frac{45°}{2} = \frac{1-\cos 45°}{\text{sen}\, 45°} = \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\frac{2-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{2-\sqrt{2}}{\sqrt{2}}$$

$$= \frac{(2-\sqrt{2})\sqrt{2}}{(\sqrt{2})^2} = \frac{2\sqrt{2}-2}{2} = \frac{2(\sqrt{2}-1)}{2} = \sqrt{2}-1$$

EJEMPLO 4. Sabiendo que $\text{sen}\,\alpha = -\frac{2}{3}$ y que $180° < \alpha < 270°$, se pide calcular de manera exacta y sin calculadora:

a) $\cos(\alpha+30°)$ b) $\text{sen}(\alpha - 45°)$ c) $\tan 2\alpha$ d) $\text{sen}\,\frac{\alpha}{2}$ e) $\cos\frac{\alpha}{2}$

a) $\cos(\alpha+30°) = \cos\alpha \cos 30° - \text{sen}\,\alpha\, \text{sen}\, 30° = \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{\sqrt{3}}{2} - \left(-\frac{2}{3}\right)\cdot\frac{1}{2} = \frac{-\sqrt{15}}{6} + \frac{2}{6} = \rightarrow$

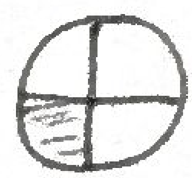
Cálculo de cos α

$\text{sen}^2\alpha + \cos^2\alpha = 1 \rightarrow \left(-\frac{2}{3}\right)^2 + \cos^2\alpha = 1 \rightarrow \frac{4}{9} + \cos^2\alpha = 1 \rightarrow \cos^2\alpha = 1 - \frac{4}{9} \rightarrow$

$\cos^2\alpha = \frac{5}{9} \rightarrow \cos\alpha = \pm\sqrt{\frac{5}{9}} = -\sqrt{\frac{5}{9}} = -\frac{\sqrt{5}}{\sqrt{9}} = -\frac{\sqrt{5}}{3}$

Se ha usado que α está en el 3° cuadrante y que por lo tanto cos α es negativo

cos −
sen −



$\to = \dfrac{-\sqrt{15}+2}{6}$

b) $\operatorname{sen}(\alpha - 45^\circ) = \operatorname{sen}\alpha \cos 45^\circ - \cos\alpha \operatorname{sen} 45^\circ = \left(-\frac{2}{3}\right)\cdot\frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{5}}{3}\right)\cdot\frac{\sqrt{2}}{2} = \frac{-2\sqrt{2}}{6} + \frac{\sqrt{10}}{6}$

$= \dfrac{-2\sqrt{2}+\sqrt{10}}{6}$

c) $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{2\cdot\frac{2\sqrt{5}}{5}}{1-\left(\frac{2\sqrt{5}}{5}\right)^2} = \dfrac{\frac{4\sqrt{5}}{5}}{1-\frac{4\cdot 5}{5^2}} = \dfrac{\frac{4\sqrt{5}}{5}}{1-\frac{4}{5}} = \dfrac{\frac{4\sqrt{5}}{\cancel{5}}}{\frac{1}{\cancel{5}}} = 4\sqrt{5}$

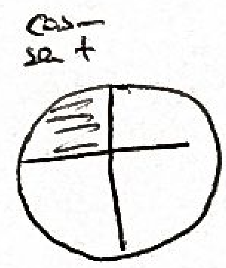
$$\boxed{\tan\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha} = \frac{-\frac{2}{\cancel{3}}}{-\frac{\sqrt{5}}{\cancel{3}}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{(\sqrt{5})^2} = \frac{2\sqrt{5}}{5}}$$

d) $\cos\frac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\alpha}{2}} = -\sqrt{\dfrac{1+\left(-\frac{\sqrt{5}}{3}\right)}{2}} = -\sqrt{\dfrac{\frac{3-\sqrt{5}}{3}}{2}} = -\sqrt{\dfrac{3-\sqrt{5}}{6}}$

Signo de $\cos\frac{\alpha}{2}$

$180^\circ < \alpha < 270^\circ \to \frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2} \to 90^\circ < \frac{\alpha}{2} < 135^\circ$

Por lo tanto $\frac{\alpha}{2}$ está en el 2° cuadrante, por lo que $\cos\frac{\alpha}{2}$ tiene signo negativo



e) $\operatorname{sen}\frac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\alpha}{2}} = \sqrt{\dfrac{1-\left(-\frac{\sqrt{5}}{3}\right)}{2}} = \sqrt{\dfrac{1+\frac{\sqrt{5}}{3}}{2}} = \sqrt{\dfrac{\frac{3+\sqrt{5}}{3}}{2}} = \sqrt{\dfrac{3+\sqrt{5}}{6}}$