1. **Extraordinaria 2025** Dadas as matrices

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & k \end{pmatrix} \text{ e } B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) Que condición ten que cumprir k para que A sexa invertible? Calcule A^{-1} cando sexa posible.

b) Para $k = 0$, calcule a matriz X que satisfai a igualdade $AX - A = B^2 + A^T$ sendo A^T a trasposta de A.

Solución:

a) A matriz ten inversa se o determinante é distinto de 0.

$$\mid A \mid = 4k + 2 - 1 - 4k = 1 \neq 0$$

A é invertible para calquera valor de k.

$$A^{-1} = \frac{(Adj(A))^t}{\mid A \mid}$$

$$Adj(A) = \begin{pmatrix} 4k-1 & -(2k-1) & -2 \\ -2k & k & -(-1) \\ 2 & -1 & k \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4k-1 & -2k & 2 \\ 1-2k & k & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

b) Resolvemos a ecuación matricial: $AX - A = B^2 + A^T \implies AX = B^2 + A^T + A$

$$B^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$AX = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 9 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$A^{-1}AX = A^{-1} \begin{pmatrix} 3 & 4 & 1 \\ 4 & 9 & 2 \\ 1 & 2 & 1 \end{pmatrix} \implies X = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \\ 4 & 9 & 2 \\ 1 & 2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 2 & 0 \\ -2 & 1 & 0 \end{pmatrix}$$

2. **Ordinaria 2025** Responda as dúas cuestións seguintes:

   a) Se $A = \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix}$ ache $\alpha, \beta \in \mathbb{R}$ tales que $A^2 + \alpha A + \beta I = 0$, onde I e 0 son as matrices identidade e cero, respectivamente.

   b) Calcule a matriz cadrada X tal que $XA = B$, se $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ e $B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Son iguais XA e AX?

Solución:

a) Calculamos A^2:

$$A^2 = \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 2 & 11 \end{pmatrix}$$

Logo a expresión que temos é:

$$\begin{pmatrix} 14 & 5 \\ 2 & 11 \end{pmatrix} + \alpha \begin{pmatrix} 2 & 5 \\ 2 & -1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 5 \\ 2 & 11 \end{pmatrix} + \begin{pmatrix} 2\alpha & 5\alpha \\ 2\alpha & -\alpha \end{pmatrix} + \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 14 + 2\alpha + \beta & 5 + 5\alpha \\ 2 + 2\alpha & 11 - \alpha + \beta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Igualando os elementos das matrices temos que $\alpha = -1$ e $\beta = -12$

b) Resolvemos a ecuación matricial: $XA = B \implies XAA^{-1} = BA^{-1} \implies X = BA^{-1}$

Temos que $|A| = 1$, logo existe matriz inversa. A calculamos cos menores e temos que:

$$A^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Logo:

$$X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Respondendo á pregunta: son iguais XA e AX. En principio non o son xa que a conmutatividade no producto de matrices non se cumpre, pero podemos comprobalo. Por un lado $XA = B$

$$AX = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Polo tanto, non son iguais XA e AX.

3. **Extraordinaria 2024** Se $A = \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix}$, dea resposta aos dous apartados seguintes:

   a) Calcule os valores de x e y que fan que A conmute con todas as matrices antisimétricas X de orde 2, é dicir, que fan que se cumpra a igualdade $AX = XA$ para toda matriz antisimétrica X de orde 2.

   b) Se $x = -1$ e $y = 1$, calcule a matriz M que satisfai a igualdade $2M = A^{-1} - AM$.

Solución:

a) A matriz antisimétrica é a que cumpre $X = -X^t$. Vexamos que ten que cumprirse nos seus termos para que se verifique isto:

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad -X^t = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$$

Para que se cumpra $X = -X^t$ temos que $a = -a$, $b = -c$, $c = -b$, $d = -d$, logo X é da forma: $X = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$

$$AX = \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix} \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} -b & b \\ -by & bx \end{pmatrix}$$

$$XA = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix} = \begin{pmatrix} bx & by \\ -b & -b \end{pmatrix}$$

Se $AX = XA$:

$$\begin{pmatrix} -b & b \\ -by & bx \end{pmatrix} = \begin{pmatrix} bx & by \\ -b & -b \end{pmatrix} \implies \begin{cases} -b = bx \\ b = by \\ -by = -b \\ bx = -b \end{cases} \implies x = -1, y = 1$$

b) $2M = A^{-1} - AM \implies 2M + AM = A^{-1} \implies (2I + A)M = A^{-1} \implies$

$M = (2I + A)^{-1}A^{-1}$

$$|A| = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \qquad Adj(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{(Adj(A))^t}{|\ A\ |} \implies A^{-1} = \frac{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}{2} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

$$2I + A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$|2I + A| = \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} = 10 \qquad Adj(2I + A) = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$$

$$(2I + A)^{-1} = \frac{(Adj(2I + A))^t}{|\ 2I + A\ |} \implies 2I + A^{-1} = \frac{\begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}}{10} =$$

$$= \begin{pmatrix} 3/10 & -1/10 \\ 1/10 & 3/10 \end{pmatrix}$$

$$M = (2I + A)^{-1}A^{-1} = \begin{pmatrix} 3/10 & -1/10 \\ 1/10 & 3/10 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/10 & -1/5 \\ 1/5 & 1/10 \end{pmatrix}$$

4. **Ordinaria 2024** Sexan A e B dúas matrices tales que $A+2B = \begin{pmatrix} 6 & -3 \\ 0 & 3 \end{pmatrix}$ e

$A+B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix}$

a) Calcula A^2.

b) Calcula a matriz X que satisfai a igualdade $A^2X - (A+B)^T = 3I - 2X$, sendo I a matriz identidade de orde 2 e $(A+B)^T$ a trasposta de $(A+B)$.

$$\underbrace{A+2B}\, - (A+B) = A+2B-A-B = B$$

$$\begin{pmatrix} 6 & -3 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} = B$$

$$A = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} - B = \begin{pmatrix} 4 & -1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

a) $$A^2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$$

b) $$A^2X - (A+B)^T = 3I - 2X$$

$$A^2X + 2X = 3I + (A+B)^T$$

$$(A^2 + 2I)X = 3I + (A+B)^T$$

$$X = (A^2+2I)^{-1}(3I + (A+B)^T)$$

$$3I + (A+B)^T = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ -1 & 5 \end{pmatrix}$$

$$A^2 + 2I = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 0 & 3 \end{pmatrix}$$

$|A^2 + 2I| = 18$ $\quad$ $Adj(A^2+2I) = \begin{pmatrix} +3 & -0 \\ -3 & +6 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -3 & 6 \end{pmatrix}$ $\quad$ $(A^2+2I)^{-1} = \dfrac{\begin{pmatrix} 3 & -3 \\ 0 & 6 \end{pmatrix}}{18} = \begin{pmatrix} 1/6 & -1/6 \\ 0 & 1/3 \end{pmatrix}$

$$X = \begin{pmatrix} 1/6 & -1/6 \\ 0 & 1/3 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 8/6 & -5/6 \\ -1/3 & 5/3 \end{pmatrix} = \begin{pmatrix} 4/3 & -5/6 \\ -1/3 & 5/3 \end{pmatrix}$$

5. **Extraordinaria 2023**

a) Calcule A se $(AB)^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ e $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

b) Se $A = \begin{pmatrix} 3 & x \\ y & z \end{pmatrix}$ é invertible, obteña os valores de x, y e z sabendo que $det(A - 3I) = 0$, que $y \neq 0$ e que $(3z)A^{-1} + I = \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix}$. Enténdase que I é a matriz identidade.

a) $(AB)^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ $\quad ((AB)^T)^T = AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$$A \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Calculamos a inversa de B

$|B| = 2$ $\quad Adj(B) = \begin{pmatrix} +1 & -(-1) \\ -1 & +1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\quad B^{-1} = \dfrac{\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}{2} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

Tamén se pode facer plantenado $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ e logo a igualdade:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \ldots$$

b) $A - 3I = \begin{pmatrix} 3 & x \\ y & z \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & x \\ y & z-3 \end{pmatrix}$

$$\left.\begin{aligned} det(A - 3I) = -xy = 0 \\ y \neq 0 \end{aligned}\right\} \Rightarrow x = 0$$

Temos que $A = \begin{pmatrix} 3 & 0 \\ y & z \end{pmatrix}$

$A = \begin{pmatrix} 3 & 0 \\ y & z \end{pmatrix}$ Imos calcular A^{-1}

$|A| = 3z \neq 0 \; (\Rightarrow z \neq 0)$ $\quad \text{Adj}(A) = \begin{pmatrix} z & -y \\ 0 & 3 \end{pmatrix}$ $\quad A^{-1} = \dfrac{\begin{pmatrix} z & 0 \\ -y & 3 \end{pmatrix}}{3z} = \begin{pmatrix} 1/3 & 0 \\ -y/3z & 1/z \end{pmatrix}$

Sabemos que: $3zA^{-1} + I = \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix}$

$$3z \begin{pmatrix} 1/3 & 0 \\ -y/3z & 1/z \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} z+1 & 0 \\ -y & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 4 \end{pmatrix} \Rightarrow \begin{cases} z+1 = 2 & \Rightarrow z = 1 \\ -y = -1 \Rightarrow y = 1 \end{cases}$$

$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$

6. **Ordinaria 2023** Despexe a matriz X da ecuación $XA = A + XB$, se A e B son matrices cadradas tales que $A - B$ é invertible. Logo calcule X se $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ e $B = (A^2 - A - 1)^{-1}$, onde I é a matriz identidade de orde 2.

$$XA = A + XB \Rightarrow XA - XB = A \Rightarrow X(A-B) = A \Rightarrow$$
$$\Rightarrow X = A(A-B)^{-1}$$

$$B = \left[\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} =$$

$$= \left[\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]^{-1} = \Big[\underbrace{-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{C} \Big]^{-1}$$

$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad |C| = 1$ $\quad \text{Adj}(C) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad C^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = B$

$A - B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad |A-B| = 2$ $\quad \text{Adj}(A-B) = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$

$(A-B)^{-1} = \dfrac{\begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}}{2} = \begin{pmatrix} 1/2 & -1 \\ 0 & 1 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ 0 & 0 \end{pmatrix}$

7. **Extraordinaria 2022**

a) Obteña a matriz antisimétrica A de orde 2×2 tal que $a_{12}=1$. Logo, calcule a súa inversa no caso de que exista. **Nota:** a_{ij} é o elemento que está na fila i e na columna j de A.

b) Sexa $A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Se $B=\begin{pmatrix} 0 & b_{12} \\ 1 & b_{22} \end{pmatrix}$, ache os valores que b_{12} e de b_{22} sabendo que B non ten inversa e que $\det(A^{-1}B+A)=-1$.

a) Autisimétrica : $A=-A^t$

$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad -A^t=\begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$ $\quad A=-A^t \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} a=-a \Rightarrow a=0 \\ \left.\begin{array}{l} b=-c \\ c=-b \end{array}\right\} \Rightarrow b=-c \\ d=-d \Rightarrow d=0 \end{cases}$

$\underbrace{A=\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad a_{12}=1}_{A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$

Calculamos a inversa:

$|A|=1 \quad Adj(A)=\begin{pmatrix} 0 & -(-1) \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A^{-1}=\dfrac{(Adj(A))^t}{|A|}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) Se non ten inversa $|B|=0$

$|B|=-b_{12}=0 \Rightarrow b_{12}=0$

$A^{-1}B+A=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & b_{22} \end{pmatrix}+\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} -1 & -b_{22} \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}=\begin{pmatrix} -1 & 1-b_{22} \\ -1 & 0 \end{pmatrix}$

$|A^{-1}B+A|=1-b_{22}=-1 \Rightarrow b_{22}=2$

$B=\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$

8. **Ordinaria 2022** Despexe X da ecuación matricial $AB(X-I)=C$, onde I é a matriz identidade (asuma que o produto AB ten inversa). Logo, calcule X se:

$$A=\begin{pmatrix}1&2&3\\0&1&0\\0&0&1\end{pmatrix}\qquad B=\begin{pmatrix}1&2&0\\0&1&0\\1&0&2\end{pmatrix}\qquad C=\begin{pmatrix}2&2&4\\0&0&1\\0&1&0\end{pmatrix}$$

$$AB(X-I)=C \Rightarrow ABX-AB=C \Rightarrow ABX=C+AB \Rightarrow$$

$$\Rightarrow X=(AB)^{-1}(C+AB)$$

$$AB=\begin{pmatrix}1&2&3\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\1&0&2\end{pmatrix}=\begin{pmatrix}4&4&6\\0&1&0\\1&0&2\end{pmatrix}$$

$$|AB|=8-6=2$$

$$Adj(AB)=\begin{pmatrix}+2&-0&+(-1)\\-8&+2&-(-4)\\+(-6)&-0&+4\end{pmatrix}=\begin{pmatrix}2&0&-1\\-8&2&4\\-6&0&4\end{pmatrix}$$

$$(AB)^{-1}=\frac{(Adj(AB))^t}{|AB|}=\frac{\begin{pmatrix}2&-8&-6\\0&2&0\\-1&4&4\end{pmatrix}}{2}=\begin{pmatrix}1&-4&-3\\0&1&0\\-1/2&2&2\end{pmatrix}$$

$$C+AB=\begin{pmatrix}2&2&4\\0&0&1\\0&1&0\end{pmatrix}+\begin{pmatrix}4&4&6\\0&1&0\\1&0&2\end{pmatrix}=\begin{pmatrix}6&6&10\\0&1&1\\1&1&2\end{pmatrix}$$

$$X=(AB)^{-1}(C+AB)=\begin{pmatrix}1&-4&-3\\0&1&0\\-1/2&2&2\end{pmatrix}\begin{pmatrix}6&6&10\\0&1&1\\1&1&2\end{pmatrix}=\begin{pmatrix}3&-1&0\\0&1&1\\-1&1&1\end{pmatrix}$$

9. **Extraordinaria 2021** Despexe X na ecuación matricial $B(X - I) = A$, onde I é a matriz identidade e A e B son matrices cadradas, con B invertible. Logo, calcule X se:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 2 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$X - I = B^{-1}A \Rightarrow X = B^{-1}A + I$$

(observa que neste caso despexei de xeito diferente ao exercicio anterior, no caso previo multipliquei o paréntesis pola matriz e neste caso xa no primer paso multiplico pola inversa, calquera das dous xeitos e correctos, mentres que teñas coidado en poñer en orde as operacións)

$$|B| = 1/6 \qquad Adj(B) = \begin{pmatrix} 1/6 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad B^{-1} = \frac{(Adj(B))^t}{|B|} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$B^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ -6 & 6 & -6 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ -6 & 6 & -6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 2 \\ -6 & 6 & -5 \end{pmatrix}$$

10. **Ordinaria 2021** Sexa $A = (a_{ij})$ a matriz de dimensión 3×3 definida por $aij = \begin{cases} 1 & \text{se} \quad i = 2 \\ (-1)^j(i-1) & \text{se} \quad i \neq 2 \end{cases}$. Explique se A e $A + I$ son ou non invertibles e calcule as inversas cando existan. (Nota: a_{ij} é o elemento de A que está na fila i e na columna j, e I é a matriz identidade.)

$$A = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 2 & -2 \end{pmatrix} \end{array} \begin{array}{l} \leftarrow \text{Fila } 1 \Rightarrow i = 1 \quad (i-1) = 0 \\ \leftarrow \text{Fila } 2 \Rightarrow i = 2 \\ \leftarrow \text{Fila } 3 \Rightarrow i = 3 \quad (i-1) = 2 \end{array}$$

Posto que a fila 1 é todo 0, o determinante vai ser 0, polo que A non é invertible

$$A + I = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{pmatrix}$$

$$|A+I| = -2 - 2 = -4 \qquad Adj(A+I) = \begin{pmatrix} +(-4) & -1 & +6 \\ -0 & +(-1) & -2 \\ +0 & -1 & +2 \end{pmatrix} = \begin{pmatrix} -4 & -1 & 6 \\ 0 & -1 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

$$(A+I)^{-1} = \frac{(Adj(A+I))^t}{|A+I|} = \frac{\begin{pmatrix} -4 & 0 & 0 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{pmatrix}}{-4} = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/4 \\ -3/2 & 1/2 & -1/2 \end{pmatrix}$$

11. **Extraordinaria 2020** Para a ecuación matircial $A^2X + AB = B$, pídese:

a) Despexar X supoñendo que A (e por tanto A^2) é invertible, e dicir cales serían as dimensión de X e de B se A tivese dimensión 4×4 e B tivese 3 columnas.

b) Resolvela no caso en que $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$ e $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$.

a) $A^2X + AB = B \Rightarrow A^2X = B - AB \Rightarrow X = (A^2)^{-1}(B - AB)$

Se A ten dimensión 4×4 e B $m\times3$, para poder multiplicar AB as dimensións de B teñen que ser 4×3

$4\times\boxed{4 \cdot 4}\times3$. As dimensións de AB sería 4×3

$(A^2)^{-1}$ terá tamén dimensión 4×4

$B-AB$ terá dimensión 4×3

A dimensión de X será 4×3

$\underset{(A^2)^{-1}}{4\times\boxed{4 \cdot 4}}\underset{B-AB}{\times 3}$

b) $AB = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -10 \end{pmatrix}$

$B - AB = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -3 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 7 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ -3 & 0 & 10 \end{pmatrix}$

$|A^2| = 10 - 9 = 1$ $\quad Adj(A^2) = \begin{pmatrix} +10 & -0 & +3 \\ -0 & +1 & -0 \\ +3 & -0 & +1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

$(A^2)^{-1} = \dfrac{(Adj(A^2))^t}{|A^2|} = \begin{pmatrix} 10 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$

$X = \begin{pmatrix} 10 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

12. **Ordinaria 2020** Sexan A e B as dúas matrices que cumpren $A+B = \begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix}$ e $A-B = \begin{pmatrix} 0 & -4 \\ 4 & -2 \end{pmatrix}$. Pídese:

a) Calcular $A^2 - B^2$. (Advertencia: neste caso, $A^2 - B^2 \neq (A+B)(A-B)$.)

b) Calcular a matriz X que cumpre a igualdade $XA + (A+B)^T = 2I + XB$, sendo I a matriz identidade de orde 2 e $(A+B)^T$ a trasposta de $A+B$.

a) $A+B+(A-B) = 2A$

$$\begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & -2 \end{pmatrix} = 2A \Rightarrow A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$$

$A+B-(A-B) = 2B$

$$\begin{pmatrix} 2 & 4 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -4 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 8 \\ -4 & 2 \end{pmatrix} = 2B \Rightarrow B = \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 8 \\ -4 & -7 \end{pmatrix}$$

$$A^2 - B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -7 & 8 \\ -4 & -7 \end{pmatrix} = \begin{pmatrix} 8 & -8 \\ 4 & 8 \end{pmatrix}$$

b) $XA + (A+B)^t = 2I + XB \Rightarrow XA - XB = 2I - (A+B)^t \Rightarrow$

$\Rightarrow X(A-B) = 2I - (A+B)^t \Rightarrow$

$\Rightarrow X = [2I - (A+B)^t](A-B)^{-1}$

$$2I - (A+B)^t = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4 & 2 \end{pmatrix}$$

$$|A-B| = 16 \qquad Adj(A-B) = \begin{pmatrix} -2 & -4 \\ 4 & 0 \end{pmatrix} \qquad (A-B)^{-1} = \frac{(Adj(A-B))^t}{|A-B|} = \begin{pmatrix} -1/8 & 1/4 \\ -1/4 & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 0 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} -1/8 & 1/4 \\ -1/4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$